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Spontaneously anisotropic spin-glass order: the four-state clock and *XY* models

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Abstract. Spin-glass ordering in conventional models normally reflects, on average, the rotational symmetry of the Hamiltonian. We demonstrate that the four-state clock model is exceptional in that the average spin-glass order is essentially collinear (twofold symmetric) despite the fourfold symmetry of the Hamiltonian. Fluctuation effects are predicted only for systems with pure states unrelated by symmetry. However, the *XY* spin glass with finite fourfold anisotropy is predicted to have conventional (fourfold symmetric/isotropic) ordering.

1. Introduction

Spin glasses continue to be of much active interest (for recent reviews see Binder and Young 1986, van Hemmen and Morgenstern 1983, 1986, Mézard *et al* 1987, Sherrington 1987) with new features and applications discovered regularly. This paper introduces another hitherto unexpected feature, an anisotropic spin-glass solution to a spin-glass model with an isotropic Hamiltonian.

The model in question is a four-state clock model with symmetric exchange $P\{J_{ij}\} = P\{-J_{ij}\}$. A p-state clock model is a special case of a vector spin model in which the vectors (of constant length) may point only in p equally angularly spaced orientations in a plane. To date, it has generally been considered that vector spin-glass models with isotropic and unbiased exchange and in the absence of external fields or anisotropy can be described in terms of spin-glass solutions which are isotropic in spin space, i.e.

$$Q^{\alpha\beta}_{\mu\nu} = N^{-1} \sum_i \langle S^{\alpha}_{i\mu} S^{\beta}_{i\nu} \rangle = Q^{\alpha\beta} \delta_{\mu\nu} \tag{1.1}$$

where α, β denote replicas (Edwards and Anderson 1975), μ, ν cartesian coordinates, and the $\langle\ \rangle$ brackets indicate a thermal average over the replica Hamiltonian. One might therefore anticipate that even-p¶ clock spin glasses with symmetric exchange distributions will exhibit a corresponding isotropy over the allowed spin orientations. In this paper we demonstrate that, although all other clock glasses exhibit this isotropy

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¶ Odd-p clocks lack inversion as an operation in their space of variables. For $p = 3$, for which clock and Potts models are isomorphic, this is known to lead to a ferromagnetic component to the spin-glass phase beneath a given critical temperature (Elderfield and Sherrington 1983a).

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in their transitions from paramagnet to spin glass†, the four-state clock is exceptional. For the four-state clock, for all temperatures beneath the paramagnet to spin-glass transition, the spin-glass state is highly anisotropic;

$$Q^{\alpha\beta}_{\mu\nu} = Q^{\alpha\beta}\delta_{\mu\nu}\delta_{\mu x}$$

or

$$Q^{\alpha\beta}_{\mu\nu} Q^{\alpha\beta}\delta_{\mu\nu}\delta_{\mu y} \tag{1.2}$$

with small fluctuations from collinearity only showing up because of replica symmetry breaking.

In § 2 we demonstrate the special character of $p = 4$ within the realm of p-state clock glasses and show, by means of a mapping to two identical Ising spin glasses and by considering the influence of an infinitesimal field, that (1.2) should follow. In § 3 we introduce a higher-order test function able to distinguish between isotropy and quasi-collinearity, and also (in principle) to show up the fluctuations from perfect collinearity expected from replica symmetry breaking. We also report Monte Carlo simulations demonstrating the collinearity.

Finally, in § 4, we discuss the relationship of the four-state clock to an XY spin glass with fourfold anisotropy. The spin-glass phase has the normal fourfold symmetry except in the limit of infinite anisotropy where there is an effectively first-order transition to twofold symmetry.

2. The four-state clock spin glass

A p-state clock model is defined by

$$H = -\sum_{(ij)} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{2.1}$$

where the $\boldsymbol{S}_i$ are unit vectors restricted to p equally angularly spaced orientations in a plane. In the infinite-range spin-glass version the summation is over all pairs (ij) with the J_{ij} quenched random couplings distributed according to the probability (Sherrington and Kirkpatrick 1975)

$$P(J_{ij}) = (N/2\pi J^2)^{1/2} \exp[-N(J_{ij} - J_0/N)^2/2J^2]. \tag{2.2}$$

In the present discussion we assume this form but we take the ferromagnetic offset J_0 to be zero, so the system is unbiased.

For the four-state model (2.1) can usefully be mapped into two identical Ising models. However, for the moment we shall not pursue this mapping but rather provide a discussion in terms applicable to a general p-state model. We do so within a representation in which the components of $\boldsymbol{S}_i$ are

$$S_{ix} = \cos\theta_i \qquad S_{iy} = \sin\theta_i \tag{2.3}$$

$$\theta_i = \frac{2\pi}{p} k_i \qquad (k_i = 0, 1, \ldots, (p-1)). \tag{2.4}$$

† Even for $p = 3$, the transition from paramagnet to spin glass is to an 'isotropic' spin-glass phase if the exchange distribution is symmetric, although it does have other unusual features when replica symmetry breaking is considered (Elderfield and Sherrington 1983a, b, Goldbart and Elderfield 1985, Gross *et al* 1985). The anisotropic spin glass occurs only at a lower temperature for symmetric exchange.

Applying the standard replica trick (Edwards and Anderson 1975, Sherrington and Kirkpatrick 1975), one obtains the free energy per spin in the thermodynamic limit ($N \to \infty$) as the extremal problem

$$\beta f = \lim_{n\to 0} \frac{1}{n} \min[g(R^\alpha, \{Q_{\mu\nu}^{(\alpha\beta)}\})]. \tag{2.5}$$

The functional $g(R^\alpha, \{Q_{\mu\nu}^{(\alpha\beta)}\})$ is given by

$$g(R^\alpha, \{Q_{\mu\nu}^{(\alpha\beta)}\}) = -\tfrac{1}{8}n(\beta J)^2 + \tfrac{1}{2}(\beta J)^2 \sum_\alpha (R^\alpha)^2 + \tfrac{1}{2}(\beta J)^2 \sum_{(\alpha\beta)} \sum_{\mu\nu} (Q_{\mu\nu}^{(\alpha\beta)})^2 - \ln \operatorname{Tr} \exp\{H_{\text{eff}}\} \tag{2.6}$$

where

$$H_{\text{eff}} = (\beta J)^2 \sum_\alpha R^\alpha[(S_x^\alpha)^2 - \tfrac{1}{2}] + (\beta J)^2 \sum_{(\alpha\beta)} \sum_{\mu\nu} Q_{\mu\nu}^{(\alpha\beta)} S_\mu^\alpha S_\nu^\beta. \tag{2.7}$$

As usual, α and β are replica labels; $\alpha, \beta = 1, \ldots, n$. The spins and trace are single site, R^α is a quadrupolar parameter given by

$$R^\alpha = \langle (S_x^\alpha)^2 \rangle - \tfrac{1}{2} \tag{2.8}$$

while $Q_{\mu\nu}^{(\alpha\beta)}$ is the usual spin-glass parameter:

$$Q_{\mu\nu}^{(\alpha\beta)} = \langle S_\mu^\alpha S_\nu^\beta \rangle \qquad \alpha \neq \beta \tag{2.9}$$

and $\Sigma_{(\alpha\beta)}$ denotes a sum over pairs of different replicas, $\alpha \neq \beta$. In each case the $\langle\ \rangle$ bracket denotes thermal averaging with respect to H_{eff}. For $p = 2$, our model reduces to the well known Ising spin glass of Sherrington and Kirkpatrick (1975) and R is zero. Below, we take $p > 2$.

Note that g satisfies the symmetry condition

$$g(R^\alpha, \{Q_{\mu\nu}^{(\alpha\beta)}\}) = g(-R^\alpha, \{Q_{\bar\mu\bar\nu}^{(\alpha\beta)}\}) \tag{2.10}$$

where $\bar\mu$, $\bar\nu$ are the complements of μ, ν, i.e. if

$$\begin{aligned} \mu &= x \qquad \text{then} \qquad \bar\mu = y \\ \mu &= y \qquad \text{then} \qquad \bar\mu = x. \end{aligned} \tag{2.11}$$

The onset of ordering of quadrupolar or spin-glass type is signalled by the change of sign of the corresponding quadratic contributions to a Landau expansion of g:

$$g(R^\alpha, \{Q_{\mu\nu}^{(\alpha\beta)}\}) = -ng_0 + \tfrac{1}{2}(\beta J)^2 \left(A_{RR} \sum_\alpha (R^\alpha)^2 + A_{QQ} \sum_{(\alpha\beta)} \sum_{\mu\nu} (Q_{\mu\nu}^{(\alpha\beta)})^2 \right) + \ldots \tag{2.12}$$

with

$$g_0 = \tfrac{1}{8}(\beta J)^2 + \ln p \tag{2.13}$$

$$A_{RR} = 1 - \tfrac{1}{8}(\beta J)^2(1 + \delta_{4,p}) \tag{2.14a}$$

$$A_{QQ} = 1 - \tfrac{1}{4}(\beta J)^2. \tag{2.14b}$$

At high temperatures A_{RR} and A_{QQ} are positive and the minimum of g has $R^\alpha = 0$, $Q_{\mu\nu}^{(\alpha\beta)} = 0$, corresponding to a paramagnet. As the temperature is lowered A_{QQ} becomes zero at

$$T_g = J/2 \tag{2.15}$$

(we work in units with $k_B = 1$), signalling the onset of spin-glass order. For $p \neq 4$, A_{RR} does not change sign until a lower temperature and the transition at T_g is to the normal isotropic spin-glass order. On the other hand, for the special case of $p = 4$, both A_{QQ} and A_{RR} becomes zero at $T = J/2$. Since R is an anisotropic order parameter this suggests that, for $p = 4$, the spin-glass phase is anisotropic. To demonstrate this clearly it is, however, useful to change representation.

We now concentrate on $p = 4$. Introducing Ising variables σ_i, τ_i $(= \pm 1)$ related to the S_i by

$$S_{ix} = \tfrac{1}{2}(\tau_i + \sigma_i) \qquad S_{iy} = \tfrac{1}{2}(\tau_i - \sigma_i) \tag{2.16}$$

the Hamiltonian of equation (2.1) may be re-expressed as

$$H = -\sum_{(ij)} \tilde{J}_{ij}(\tau_i\tau_j + \sigma_i\sigma_j) \qquad \{\tilde{J}_{ij}\} = \tfrac{1}{2}\{J_{ij}\} \tag{2.17}$$

i.e. the four-state clock model is equivalent to two independent Ising models with identical exchange interactions of strength one-half of those of the original clock model. We can thus use our knowledge of the Ising spin glass to analyse the four-state clock glass.

Let us note the relation of the simultaneous mode-softening of both Q and R degrees of freedom to the Ising model. In a replication of the Ising system described by equation (2.17) there are $2n$ spins per site, leading to the softening at the transition of $2n(2n-1)/2 = (2n^2 - n)$ Ising $q^{(\alpha\beta)}$ modes. In the clock representation there are n spins with four combinations, xx, xy, yx, yy, giving $4n(n-1)/2 = (2n^2 - 2n)$ $Q^{(\alpha\beta)}$ modes. Including the n R^α modes gives a total of $(2n^2 - n)$ modes, as in the Ising representation. This argument confirms that the R^α modes are completely equivalent to the $Q^{(\alpha\beta)}$ modes for the four-state clock and must therefore be included in any description of the ordered state.

We may now give a simple argument in terms of the effects of an infinitesimal symmetry breaking field $\boldsymbol{h}$, adding to the Hamiltonian in equation (2.1) the term

$$H_{\text{field}} = -\boldsymbol{h} \cdot \sum_i \boldsymbol{S}_i = -h_x \sum_i S_{ix} - h_y \sum_i S_{iy}. \tag{2.18}$$

If the spin-glass phase is isotropic it should be unaffected by the application of $\boldsymbol{h}$ in the limit $h \to 0$. We shall see that this is not the case. In terms of the Ising variables

$$H_{\text{field}} = -h_\tau \sum_i \tau_i - h_\sigma \sum_i \sigma_i \tag{2.19}$$

where

$$h_\tau = \tfrac{1}{2}(h_x + h_y) \qquad h_\sigma = \tfrac{1}{2}(h_x - h_y). \tag{2.20}$$

Consider the situation where $h_x > |h_y| > 0$, so that both h_τ and h_σ are positive†. Then one has,

$$\operatorname{sgn}\langle\tau_i\rangle_T = \operatorname{sgn}\langle\sigma_i\rangle_T. \tag{2.21}$$

If, furthermore, h_y tends to zero then one has effective equivalence between the τ and σ so that

$$\langle\tau_i\rangle_T = \langle\sigma_i\rangle_T \tag{2.22}$$

† It is interesting to note that, in the presence of a field with $h_x \neq h_y$, the four-state clock model will exhibit two Almeida–Thouless replica symmetry breaking transitions (corresponding to the τ and σ transitions determined, respectively, by h_τ and h_σ).

and

$$\langle S_{iy}\rangle_T = 0 \tag{2.23}$$

where $\langle\ \rangle_T$ denotes the average with respect to H. Similarly for $h_y > |h_x| > 0$ in the limit $h_x \to 0$ one has

$$\langle \tau_i\rangle_T = -\langle \sigma_i\rangle_T \tag{2.24}$$

so

$$\langle S_{ix}\rangle_T = 0. \tag{2.25}$$

In each situation the larger field needs only to be infinitesimal, so that in either case, an infinitesimal field suffices to orient all the spins along the same axis on average over all the thermodynamic states. We refer to this cooperative order as collinear.

Note that the above argument is independent of the details of $\{J_{ij}\}$. In particular, it is independent of whether there are many pure states not related by symmetry, as is the case for the infinite-range Sherrington–Kirkpatrick model, or only pure states related by global symmetry operations, as argued to be the case for short-ranged spin glasses of appropriate spatial dimensionality (Fisher and Huse 1986, Bray and Moore 1986). However, as indicated for example by replica symmetry breaking, there can be significant fluctuations from average collinearity, as we demonstrate explicitly in the next section. An alternative discussion, based on thermodynamic state representations, is given in the appendix.

3. Monte Carlo simulation

We have carried out computer simulation to test our assertion of anisotropic spin-glass order. Since we must simulate a finite system, we required that a test quantity

(i) preserves the symmetry of the Hamiltonian (inversion and $x \leftrightarrow y$ interchange), and

(ii) provides different results for isotropic and anisotropic order.

Such a quantity is

$$\Psi = \frac{4[(\langle Q_{xx}^2 Q_{yy}^2\rangle_T + \langle Q_{xy}^2 Q_{yx}^2\rangle_T)]_{\mathrm{av}}}{[\langle Q_{xx}^2 + Q_{yy}^2 + Q_{xy}^2 + Q_{yx}^2\rangle_T^2]_{\mathrm{av}}} \tag{3.1}$$

where

$$Q_{\mu\nu} = N^{-1} \sum_i S_{i\mu}^{(1)} S_{i\nu}^{(2)} \qquad \mu, \nu = x, y. \tag{3.2}$$

$\{S^{(1)}\}$ and $\{S^{(2)}\}$ are two independent replicas of the system, the $\langle\ \rangle_T$ refer to thermal averages/Monte Carlo averages over the two independently evolving systems, and $[\ ]_{\mathrm{av}}$ refers to an average over different realisations of the $\{J_{ij}\}$.

In the thermodynamic limit and for $T > T_g$, the $Q_{\mu\nu}$ have independent Gaussian distributions with zero mean, so $\Psi = \frac{1}{2}$. Were the low-temperature order isotropic, then for $T \to 0$, $\lim_{N\to\infty} \Psi = 1$, whereas perfect collinearity would give $\lim_{N\to\infty} \Psi = 0$ for $T < T_g$.

In the infinite-range case we can again use the Ising mapping to relate to the order parameter space of the Sherrington–Kirkpatrick model. From the relations

$$[\langle Q_{xx}^2+Q_{yy}^2+Q_{xy}^2+Q_{yx}^2\rangle_T^2]_{av}=N^{-4}\sum_{ijkl}[\langle\sigma_i\sigma_j\rangle_T^2\langle\sigma_k\sigma_l\rangle_T^2]_{av} \tag{3.3}$$

$$\begin{aligned}[\langle Q_{xx}^2Q_{yy}^2+Q_{xy}^2Q_{yx}^2\rangle_T]_{av} &= (128N^4)^{-1}\sum_{ijkl}\{4[\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle_T^2]_{av}+12[\langle\sigma_i\sigma_j\rangle_T^2\langle\sigma_k\sigma_l\rangle_T^2]_{av}\\ &\quad -8[\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle_T\langle\sigma_i\sigma_j\rangle_T\langle\sigma_k\sigma_l\rangle_T]_{av}\\ &\quad -8[\langle\sigma_i\sigma_j\rangle_T\langle\sigma_j\sigma_k\rangle_T\langle\sigma_k\sigma_l\rangle_T\langle\sigma_l\sigma_i\rangle_T]_{av}\}\end{aligned} \tag{3.4}$$

we find that, in the thermodynamic limit, for $T<T_g$, Ψ may be expressed in terms of the Parisi function $q(x)$ (Parisi 1979) as

$$\Psi=F/G \tag{3.5}$$

where

$$\begin{aligned}F=\frac{1}{8}\bigg(&\langle q^4\rangle+3\langle q^2\rangle^2-4\langle q\rangle^2\langle q^2\rangle+\int_0^1 dx\int_0^x dy[q^2(x)-q^2(y)]^2\\ &-4\langle q\rangle\int_0^1 dx\,q(x)\int_0^x dy[q(x)-q(y)]^2\\ &-\int_0^1 dx\int_0^x dy\int_0^x dz[q(x)-q(y)]^2[q(x)-q(z)]^2\bigg)\end{aligned} \tag{3.6}$$

$$G=-\langle q^4\rangle+4\langle q^2\rangle^2+2\int_0^1 dx\int_0^x dy[q^2(x)-q^2(y)]^2 \tag{3.7}$$

and

$$\langle q^m\rangle=\int_0^1 dx\,q^m(x). \tag{3.8}$$

Clearly, if $q(x)$ is a non-zero constant, the replica symmetric situation for $T<T_g$, this indeed leads to $\Psi=0$. More generally, for the replica symmetry broken situation this result holds only in the limits $T\to T_g^-$ and $T\to 0$. For small $\tau=(T_g-T)/T_g$, $q(x)$ is given by (Kondor 1983, Thomsen *et al* 1986, Sommers 1985)

$$\begin{aligned}q(x)&=\tfrac{1}{2}(1+3\tau)x+O(\tau^3) && x<x_1\\ &=q(1) && x>x_1\end{aligned} \tag{3.9}$$

with

$$q(1)=\tau+\tau^2-\tau^3+O(\tau^4) \tag{3.10}$$

$$x_1=2\tau-4\tau^2+O(\tau^3) \tag{3.11}$$

leading to

$$\Psi(\tau)=\tfrac{4}{45}\tau(1-\tfrac{76}{45}\tau)+O(\tau^3) \tag{3.12}$$

so $\Psi\to 0$ as $T\to T_g^-$. Also, as $T\to 0$, $q(x)\to 1$ so Ψ vanishes in this limit too. Thus, Ψ is expected to be zero at $T=T_g^-$ and $T=0$, and non-zero in between, but remaining relatively small.

To test the above argument a Monte Carlo simulation of Ψ was performed, using the original clock representation and a 'heat bath' Monte Carlo algorithm (Binder 1979). The thermal averages were performed as Monte Carlo averages over times ranging from t_0 to nt_0, $n > 2$, where t_0 is the equilibration time estimated by the method of Bhatt and Young (1985), viz looking for the coalescence of upper and lower bounds to the spin-glass susceptibility:

$$\chi_{SG} = N^{-1} \sum_{ij} \left[\left\langle \sum_{\mu} S_{i\mu} S_{j\mu} \right\rangle_T^2 \right]_{av}. \tag{3.13}$$

The upper bound is determined by

$$\chi_{SG}^{U}(t) = N^{-1} \left[\sum_{\mu\nu} \left(\sum_{i} S_{i\mu}(t_0) S_{i\nu}(t+t_0) \right)^2 \right]_{av} \tag{3.14}$$

involving temporal correlations of a single system, averaged over many realisations of the $\{J_{ij}\}$. The lower bound involves correlations between two identical but simultaneously evolving systems 1, 2, given by

$$\chi_{SG}^{L}(t) = N^{-1} \left[\left\langle \sum_{\mu\nu} \left(\sum_{i} S_{i\mu}^{(1)}(t+t_0) S_{i\nu}^{(2)}(t+t_0) \right)^2 \right\rangle_T \right]_{av} \tag{3.15}$$

with the two systems initially uncorrelated. Data were only accepted if $\chi_{SG}^{U}(t_0)$ and $\chi_{SG}^{L}(t_0)$ agreed within the errors.

Naturally, only finite-sized systems can be studied, but a study of several different sizes can reveal key features. Our results for various sizes and temperatures are shown in figure 1. They are clearly in accord with the collinear/anisotropic prediction of a step function at $T_g = J/2$ from $\Psi = 0$ at T_g^- to $\Psi = \frac{1}{2}$ for $T > T_g$ in the thermodynamic limit, $N \to \infty$. They unequivocally rule out isotropy. We were unable to detect the rise beneath T_g^- predicted by the Parisi theory, but this may be because the sizes studied are too small.

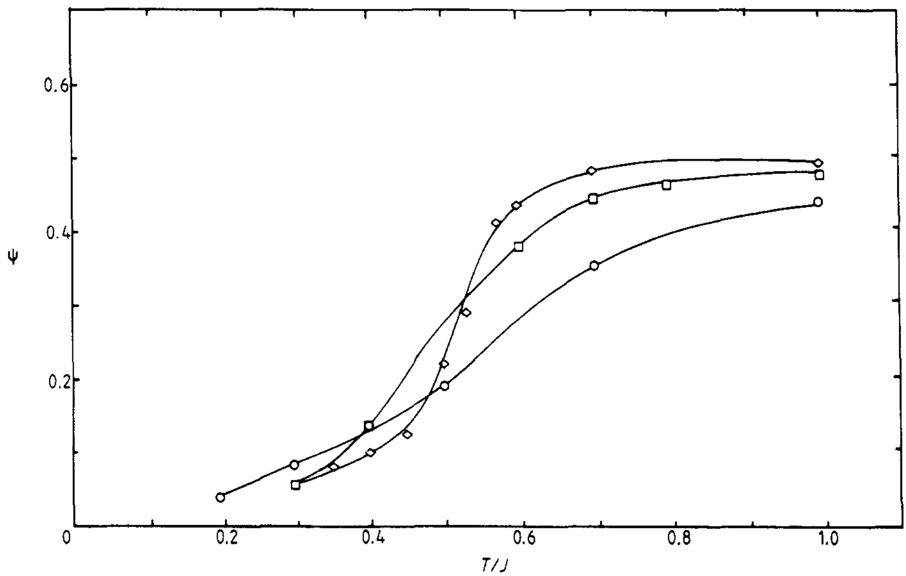

Figure 1. Plot of Ψ against T/J for $N = 20$ (○), 80 (□), 320 (◇). The transition temperature is predicted as $T_g/J = 0.5$.

4. *XY* model with fourfold anisotropy

Since we have argued that the four-state clock glass is 'collinear', it is interesting to investigate the role of fourfold anisotropy on the XY model and look at how the glass phase behaves as we change the strength of this anisotropy. Thus we consider the model defined by the Hamiltonian

$$H = -\sum_{(ij)} J_{ij} \cos(\theta_i - \theta_j) - D \sum_i \cos 4\theta_i \tag{4.1}$$

where the θ_i are continuous angles varying from 0 to 2π and the $\{J_{ij}\}$ obey the same probability distribution as in equation (2.2) with $J_0 = 0$. Clearly, $D = 0$ and $D = \infty$ correspond respectively to the pure XY and four-state clock limits. By an appropriate choice of origin of θ, D may be assumed non-negative.

Standard analysis leads to an expression for the free energy functional g as in equation (2.12) but now with

$$g_0 = \tfrac{1}{8}(\beta J)^2 + \ln[I_0(\beta D)] \tag{4.2a}$$

$$A_{RR} = 1 - \tfrac{1}{8}(\beta J)^2(1+\Delta) \tag{4.2b}$$

where

$$\Delta = I_1(\beta D)/I_0(\beta D) \tag{4.2c}$$

and the $I_k(\beta D)$ denote modified Bessel functions of the first kind of order k. For D large we can expand the Bessel functions

$$\Delta = 1 - \frac{1}{2\beta D} - \frac{1}{8(\beta D)^2} + \mathrm{O}((\beta D)^{-3}). \tag{4.3}$$

Thus, for D finite we always have $\Delta < 1$ and the quadrupolar parameter critical temperature, associated with $A_{RR} = 0$, is always lower than the Q-ordering temperature $T_g = J/2$†, associated with $A_{QQ} = 0$, resulting in fourfold symmetric spin-glass order for T just below T_g. If the renormalising effect of the fourfold symmetric spin-glass ordering were ignored, a second transition to a partially collinear state would occur at the lower temperature where $A_{RR} = 0$. However, when such effects are included, modifications result, lowering (or possibly removing) the second transition temperature (Toulouse and Gabay 1981).

In order to investigate whether such a transition from fourfold to twofold symmetric order occurs, we consider further the stability properties of g against fluctuations breaking the fourfold symmetry. To this end we define

$$Q^{(\alpha\beta)} = \frac{Q_{xx}^{(\alpha\beta)} + Q_{yy}^{(\alpha\beta)}}{2} \qquad Z^{(\alpha\beta)} = \frac{Q_{xx}^{(\alpha\beta)} - Q_{yy}^{(\alpha\beta)}}{2} \qquad (\alpha \neq \beta) \tag{4.4}$$

and assume that the crossed spin-glass parameters are zero ($Q_{xy}^{(\alpha\beta)} = Q_{yx}^{(\alpha\beta)} = 0$). Expanding the free-energy functional to second order in R, Z, for arbitrary $\{Q^{(\alpha\beta)}\}$,

$$g(\{R^a\}, \{Q^{(\alpha\beta)}\}, \{Z^{(\alpha\beta)}\}) = g(0, \{Q^{(\alpha\beta)}\}, 0) + \tfrac{1}{2}(\beta J)^2 \sum_{ab} S^{ab}(\{Q^{(\alpha\beta)}\})\eta^a\eta^b + \ldots \tag{4.5}$$

† Note that T_g is independent of D through the corresponding independence of A_{QQ}. In turn this is a consequence of the independence of D of the ratio of integrals

$$\int_0^{2\pi} \mathrm{d}\theta \cos^2\theta \exp(\beta D \cos 4\theta) \Big/ \left(\int_0^{2\pi} \mathrm{d}\theta \exp(\beta D \cos 4\theta) \right)^{-1}.$$

where $\boldsymbol{\eta}$ is an $\frac{1}{2}n(n+1)$-dimensional vector of elements

$$\begin{aligned} \eta^a &= R^\alpha \qquad a = 1, \ldots, n \\ \eta^a &= Z^{(\alpha\beta)} \qquad a = n+1, \ldots, \tfrac{1}{2}n(n+1) \end{aligned} \tag{4.6}$$

and $S(\{Q^{(\alpha\beta)}\})$ is the corresponding stability matrix (de Almeida and Thouless 1978). S is straightforwardly expressible in terms of correlations of the isotropic spin glass (Nobre 1989). Continuous phase transitions to lower symmetry are signalled by the softening to zero of an eigenvalue of S, with $\{Q^{(\alpha\beta)}\}$ determined so as to minimise the contribution to f in equation (2.5) of $g(0, \{Q^{(\alpha\beta)}\}, 0)$, corresponding to the 'isotropic' phase.

Within the replica symmetric approximation the stability can be expressed, in the limit $n \to 0$, in terms of

$$B_{RR} = \frac{1}{(\beta J)^2} \lim_{n\to 0} \frac{1}{n} \frac{\partial^2 g}{\partial R^2} = 1 - (\beta J)^2 [\langle c^4 \rangle_0 - \langle c^2 \rangle_0^2]_{x,y} \tag{4.7a}$$

$$\begin{aligned} B_{ZZ} &= -\frac{2}{(\beta J)^2} \lim_{n\to 0} \frac{1}{n} \frac{\partial^2 g}{\partial Z^2} \\ &= 2 - (\beta J)^2 [\langle c^2 \rangle_0^2 - 2\langle cs \rangle_0^2 + \langle s^2 \rangle_0^2 - 4\langle c^2 \rangle_0 \langle c \rangle_0^2 + 8\langle cs \rangle_0 \langle c \rangle_0 \langle s \rangle_0 - 4\langle s^2 \rangle_0 \langle s \rangle_0^2 \\ &\quad + 3\langle c \rangle_0^4 - 6\langle c \rangle_0^2 \langle s \rangle_0^2 + 3\langle s \rangle_0^4]_{x,y} \end{aligned} \tag{4.7b}$$

and

$$\begin{aligned} B_{RZ} &= -\frac{1}{(\beta J)^2} \lim_{n\to 0} \frac{1}{n} \frac{\partial^2 g}{\partial R \partial Z} \\ &= -(\beta J)^2 [\langle c \rangle_0 \langle c^3 \rangle_0 - \langle c^2 \rangle_0 \langle c \rangle_0^2 - \langle s \rangle_0 \langle c^2 s \rangle_0 + \langle c^2 \rangle_0 \langle s \rangle_0^2]_{x,y}. \end{aligned} \tag{4.7c}$$

In the equations above

$$c \equiv \cos\theta \qquad s \equiv \sin\theta \tag{4.8}$$

$\langle\ \rangle_0$ denotes an average over $\theta(0, 2\pi)$ with probability proportional to $\exp(-\beta H_0)$ where

$$H_0 = -D\cos 4\theta + JQ^{1/2}(x\cos\theta + y\sin\theta) \tag{4.9}$$

and $[\ ]_{x,y}$ stands for an average over x and y which are independent Gaussian random variables with zero mean and unit variance. Q is given self-consistently by

$$Q = [\langle c \rangle_0^2]_{x,y} = [\langle s \rangle_0^2]_{x,y}. \tag{4.10}$$

The condition for stability of the fourfold symmetric phase against replica symmetric fluctuations lowering the symmetry to twofold is that the corresponding eigenvalues $\lambda_{1,2}$ of S are non-negative, which is equivalent to

$$\delta = \lambda_1 \lambda_2 = B_{RR} B_{ZZ} + 2B_{RZ}^2 \geqslant 0 \tag{4.11a}$$

$$\gamma = \lambda_1 + \lambda_2 = B_{RR} + B_{ZZ} \geqslant 0. \tag{4.11b}$$

Expanding perturbatively for temperatures less than the paramagnetic to fourfold symmetric spin-glass transition temperature yields

$$\delta = 2(1-\Delta)[\tau - \tfrac{1}{6}(6 - 3\Delta - \Delta^2)\tau^2 + O(\tau^3)] \tag{4.12a}$$

$$\gamma = \tfrac{1}{2}(1-\Delta) + (3-\Delta)\tau + \tfrac{1}{4}(-5 + 20\Delta + \tfrac{11}{3}\Delta^2)\tau^2 + O(\tau^3) \tag{4.12b}$$

where

$$\tau = (T_g - T)/T_g \qquad T_g = J/2 \tag{4.13}$$

so that both δ and γ are greater than zero for any $D < \infty$ ($\Delta < 1$) to the order exhibited ($O(\tau^2)$). Numerical investigations of δ indicate that its stability continues for all temperatures beneath T_g. In the four-state clock limit ($D = \infty$), B_{RR}, B_{ZZ} and B_{RZ} can be evaluated exactly in terms of Q, to yield

$$\delta = 2[1 - (T_g/T)(1-Q)^2]^2 \tag{4.14a}$$

$$\gamma = 3 - 2(T_g/T)^2(1-Q)^2 \tag{4.14b}$$

which are both positive for $T < T_g$. Thus the fourfold symmetric spin-glass solution is stable against small replica symmetric fluctuations at all D, $T < T_g$.

The clock limit, $D = \infty$, is therefore particularly interesting and special. The 'collinear' and 'isotropic' spin-glass states have the *same* free energy and each is stable against small replica symmetric fluctuations. However, an infinitesimal field (or an infinitesimal twofold anisotropy) favours collinear ordering. Thus, one has a special kind of first-order transition to collinear order for $D = \infty$ only.

5. Conclusion

We have shown that the spin-glass phase of a four-state clock spin glass has essentially collinear order. Even for a symmetric exchange distribution and with only an infinitesimal field, the spin-glass phase has a lower (twofold) symmetry than that of the controlling Hamiltonian (fourfold).

A further consideration of an XY (planar) model with a fourfold anisotropy field indicates that, for a symmetric exchange distribution, the spin-glass order has the full fourfold symmetry *except* in the clock limit (infinite fourfold anisotropy) where there is a first-order transition of a novel kind to twofold symmetry.

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Appendix. Pure-state analysis of the four-state clock model

Let us now consider the four-state clock in terms of the pure thermodynamic states of the Ising model. We identify two situations for the Ising system.

(i) There are only two pure states, which are global inverses of one another. This is the situation for a conventional Ising ferromagnet, and has been argued to be the situation for a short-range Ising spin glass (e.g. Fisher and Huse 1986, Bray and Moore 1986), although that problem remains incompletely solved.

(ii) There are many pure states, which are unrelated by global symmetry (as well as global inverse pairs). This is the situation for the infinite-range spin-glass model of Sherrington and Kirkpatrick (1975). Explicit results for this case have been given in § 3.

In case (i) it is immediately clear that all pure states of the four-state clock are perfectly collinear. One has

$$\langle \tau_i \rangle_T = \pm \langle \sigma_i \rangle_T \qquad \text{all } i \tag{A1}$$

depending upon whether the τ and σ systems are in the same pure Ising state or in global inverses; an arbitrarily small field will determine these states. The former leads to

$$\langle S_{ix} \rangle_T = \langle \tau_i \rangle_T \qquad \langle S_{iy} \rangle_T = 0 \tag{A2a}$$

and the latter leads to

$$\langle S_{iy} \rangle_T = \langle \tau_i \rangle_T \qquad \langle S_{ix} \rangle_T = 0. \tag{A2b}$$

In case (ii), it is only if the τ and σ systems belong to global inverse Ising states that perfect collinearity results. In general this is not the case. Nevertheless, there does result average collinearity in the sense discussed in § 2, with fluctuations manifesting themselves only in higher-order moments, as discussed in § 3. To see this within the pure state language, let us first separate the complete set of pure states into two groups, each group having a common sign of overall magnetisation or positive overlap

$$q_{ss'} = N^{-1} \sum_i \langle \tau_i \rangle_s \langle \tau_i \rangle_{s'} \tag{A3}$$

where subscripts s, s' label the pure states; this is the normal separation into two groups of states which are effectively non-communicating in the spin-glass phase of the Ising spin glass and is the analogue of the spin-up/spin-down separation in a ferromagnet†. We restrict ourselves to one such group and for definiteness take the same group for the τ and σ systems (we could equally well take the opposite group for each).

Consider now

$$Q_{\mu\nu} = \sum_{\tilde{s},\tilde{s}'} \tilde{P}_{\tilde{s}} \tilde{P}_{\tilde{s}'} N^{-1} \sum_i \langle S_{i\mu} \rangle_{\tilde{s}} \langle S_{i\nu} \rangle_{\tilde{s}'} \qquad \mu, \nu = x, y \tag{A4}$$

where the $\tilde{s}$, $\tilde{s}'$ are states of the clock system, with probabilities $\tilde{P}_{\tilde{s}}$, $\tilde{P}_{\tilde{s}'}$. Each state $\tilde{s}$ is made up of two states, s, s' of the Ising τ and σ systems, with

$$\tilde{P}_{\tilde{s}} = P_s P_{s'} \tag{A5}$$

where P_s is the probability of state s. Noting that

$$q = \sum_{s,s'} P_s P_{s'} N^{-1} \sum_i \langle \eta_i \rangle_s \langle \xi_i \rangle_{s'} \tag{A6}$$

takes the same value irrespective of whether η, ξ are τ, σ, collinearity follows in the sense that

$$Q_{xx} = q \qquad Q_{yy} = Q_{xy} = Q_{yx} = 0. \tag{A7}$$

† Infinitesimal fields can be used to nucleate these groups but are usually considered implicitly.

Explicitly

$$Q_{\mu\nu} = \tfrac{1}{4} \sum_{s,s',s'',s'''} P_s P_{s'} P_{s''} P_{s'''} N^{-1} \sum_i \{\langle\tau_i\rangle_s \langle\tau_i\rangle_{s''} + \lambda_\mu \langle\sigma_i\rangle_{s'} \langle\tau_i\rangle_{s''} + \lambda_\nu \langle\tau_i\rangle_s \langle\sigma_i\rangle_{s'''} + \lambda_\mu \lambda_\nu \langle\sigma_i\rangle_{s'} \langle\sigma_i\rangle_{s'''}\}$$

$$= \tfrac{1}{4} q(1+\lambda_\mu)(1+\lambda_\nu) \tag{A8}$$

where

$$\lambda_\mu = \begin{cases} 1 & \mu = x \\ -1 & \mu = y. \end{cases} \tag{A9}$$

On the other hand, for case (ii), the effects of fluctuations manifest themselves in higher moment averages, such as Ψ, employed in § 3.

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